ISSN 2229-5518

Analysis of the antiplane shear of certain materials

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Abstrack
In this paper, the problem of axial shear of a hollow circular cylinder, composed of an elastic, homogeneous, isotropic material, is described. The inner surface of the tube is bonded to a rigid cylinder while the outer surface is subjected to axial shear. From some examples of energy functions, conditions on shear are set. These conditions are finally generalized for a certain class of potentials.

Keywords: Shear, elastic material, tubular structure, energy potential, non-linearity, partial differential equation.

1. Introduction

The study of shearing of elastic materials, incompressible has always been the subject of special attention in mechanics [1]. In fracture mechanics, for example, antiplan shear has been of particular interest.
Simple shear deformations, for which the displacement gradient is constant, are sustainable both in the linear and nonlinear theory. Necessary and sufficient conditions on the strain energies for homogeneous isotropic nonlinear elastic materials which do allow antiplane shear were obtained in Knowles for further contributions in the compressible case [2].
This is the case for example of the study on the propagation of a crack. In brittle fracture mechanics, the solution of the antiplan problem allows to know the crack front response. For analytical solutions in antiplan mode (or mode III in fracture mechanics), with boundary conditions equivalent to those of a linear elasticity problem, we have either regular fields or strong discontinuity lines of the gradient of the shifting. For the antiplanar shear, some authors [3] had to precede to a classification of the materials likely to undergo such a deformation. Other authors [4] have shown that this characterization of materials is closely related to the nature and form of the energy function. This characterization remains less obvious in nonlinear elasticity. In the case of telescopic shear where radial deformation is neglected, only incompressible materials are considered [5], and the study of boundary problems leads to analytical solutions. The intent of this expository paper is to draw the attention of the applied mathematics to an interesting two-dimensional mathematical model arising in solid mechanics involving a single second-order nonlinear partial differential equation. Anti-plane shear deformations are one of the simplest classes of deformations that solids can undergo. In longitudinal shear of cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. Generalized shear, with just a single scalar axial displacement field, may be viewed as complementary to the more complicated plane strain deformation, with its two in-plane displacements.
In this paper, after the formulation of the problem, we are interested in the necessary conditions for a material to undergo an axial shear on the one hand, and on the other, for the case of a certain class of materials whose shearing conditions depend on strongly of the nature of the energy density. We will end up with a more general formulation.

International Journal of Scientific & Engineering Research Volume 9, Issue 3, March-2018 667
ISSN 2229-5518

2. Basic equations

The geometric domain is a hollow cylinder composed of an elastic, isotropic material with an inner surface bounded by a rigid cylinder and an outer surface subjected to axial shear.

In a cylindrical coordinate system, consider a point M which, in the undistorted configuration has the components (R, Θ, Z) and the deformed configuration (r, θ, z).

The kinematics of deformation is described by:

$$r = r(R),\ \theta = \Theta,\ z = Z + w(R), \tag{2.1}$$

which translates for axial shear, a combined deformation of the tube: radial with $r(R)$ and longitudinal or shear anti plan with $w(R)$.

With clearly defined boundary conditions on the inner R_i and outer R_e radius [5], these two functions are solutions of a system of nonlinear differential equations. The resolution of these equations strongly depends on the shape of the energy function $W(I_1, I_2, I_3)$, where the first three invariants of the Cauchy Green tensor are $I_i, (i = 1,2,3)$.

According to (2.1), the deformation gradient tensor and the left Cauchy-Green tensor are written:

$$\mathbf{F} = \begin{bmatrix} \dot{r} & 0 & 0 \\ 0 & \dfrac{r}{R} & 0 \\ \dot{w} & 0 & 1 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} \dot{r}^2 & 0 & \dot{r}\dot{w} \\ 0 & \dfrac{r^2}{R^2} & 0 \\ \dot{r}\dot{w} & 0 & 1+\dot{w}^2 \end{bmatrix}. \tag{2.2}$$

The first three elementary invariants of $\mathbf{B}$ give:

$$\begin{aligned} I_1 &= tr(\mathbf{B}) = \dot{r}^2 + (r/R)^2 + 1 + \dot{w}^2, \\ I_2 &= tr(\mathbf{B}^*) = \dot{r}^2 + (r/R)^2(1+\dot{w}^2) + (R/r)^2, \\ I_3 &= \det(\mathbf{B}) = (r\dot{r}/R)^2, \end{aligned} \tag{2.3}$$

where $B^* = (\det B)B^{-1}$ is the adjoint of $\mathbf{B}$.

The stress tensor of Cauchy is given by [6],

$$\sigma = \beta_0 \mathbf{1} + \beta_1 \mathbf{B} + \beta_{-1} \mathbf{B}^{-1}, \tag{2.4}$$

where $\mathbf{1}$ is the identity tensor and $\beta_j, (j = 0, 1, -1)$ are given by:

$$\begin{aligned} \beta_0 &= 2I_3^{-1/2}\left[I_2 W_2 + I_3 W_3\right], \\ \beta_1 &= 2I_3^{-1/2} W_1, \\ \beta_{-1} &= -2I_3^{1/2} W_2, \end{aligned} \tag{2.5}$$

and $W_i = \partial W / \partial I_i,\ (i = 1, 2, 3)$.

In the absence of volume forces, the equilibrium equation is obtained by:

$$\mathrm{div}(\sigma) = 0. \tag{2.6}$$

What is reduced according to (2.4) to the system:

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \\ &\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} = 0. \end{aligned} \tag{2.7 b}$$

By choosing as a condition to the limits on the inside R_i [5] and outside R_e [7] radius of the tube:

$$\begin{aligned} &r(R_i) = R_i,\ \ w(R_i) = 0, \\ &\sigma_{rr}(R_e) = 0,\ \ \sigma_{rz}(R_e) = \sigma_0 \end{aligned} \tag{2.8 b}$$

with σ_0 a constant, the system (2.7) admits two unique solutions in $r(R)$ and $w(R)$.

Considering equation (2.7 b), we find that it can still be written in the form:

$$\frac{\partial}{\partial r}(r\sigma_{rz}) = 0. \tag{2.9}$$

With $\sigma_{rz} = 2W_1 I_3^{-1/2}\dot{r}\dot{w} + 2W_2 I_3^{1/2}\dfrac{\dot{w}}{\dot{r}}$, and from the expression of I_3 in (2.3), equation (2.9) becomes:

$$\frac{\partial}{\partial r}\left(\dot{w}\left(RW_1 + \frac{r^2}{R}W_2\right)\right) = 0. \tag{2.10}$$

By applying the string rule to (2.7 a); $\dfrac{\partial \sigma_{rr}}{\partial R}\dfrac{\partial R}{\partial r} + \dfrac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0$, and considering the expressions of

$\sigma_{rr} = \beta_0 + \beta_1 \dot{r}^2 + \beta_{-1} \frac{1+\dot{w}^2}{\dot{r}}$ and

$\sigma_{\theta\theta} = \beta_0 + \beta_1 \frac{r^2}{R^2} + \beta_{-1} \frac{R^2}{r^2}$, starting from (2.4) and taking into account (2.5), equation (2.7 a) can still be written:

$$\frac{\partial}{\partial R}\left[\dot{r}\left(\frac{R}{r}+\frac{r}{R}\right)W_2 + \dot{r}\frac{r}{R}W_3 + \dot{r}\frac{R}{r}W_1\right]+$$
$$\left(\dot{r}^2\frac{R}{r^2}-\frac{1}{R}\right)W_1 + \left(\dot{r}^2\frac{R}{r^2}-\frac{1}{R}\left(1+\dot{w}^2\right)\right)W_2 = 0.$$

3. Necessary shearing antiplanar condition

Assuming $r = R$, we have $I_3 = 1$, the transformation then becomes incompressible and equations (2.10) and (2.11) become:

$$\frac{\partial}{\partial R}\left[R\dot{w}\left(W_1+W_2\right)\right]=0,$$
$$\frac{\partial}{\partial R}\left(W_1+2W_2+W_3\right)-\frac{\dot{w}^2}{R}W_2 = 0.$$

On the other hand, we will have $I_1 = I_2 = 3+\dot{w}^2$, which give $\frac{\partial I_1}{\partial R} = \frac{\partial I_2}{\partial R} = 2\ddot{w}\dot{w}.$

$$\frac{\partial W_1}{\partial R} = \frac{\partial W_1}{\partial I_1}\frac{\partial I_1}{\partial R} + \frac{\partial W_1}{\partial I_2}\frac{\partial I_2}{\partial R} = 2\ddot{w}\dot{w}\left(W_{11}+W_{12}\right),$$
$$\frac{\partial W_2}{\partial R} = \frac{\partial W_2}{\partial I_1}\frac{\partial I_1}{\partial R} + \frac{\partial W_2}{\partial I_2}\frac{\partial I_2}{\partial R} = 2\ddot{w}\dot{w}\left(W_{12}+W_{22}\right),$$
$$\frac{\partial W_3}{\partial R} = \frac{\partial W_3}{\partial I_1}\frac{\partial I_1}{\partial R} + \frac{\partial W_3}{\partial I_2}\frac{\partial I_2}{\partial R} = 2\ddot{w}\dot{w}\left(W_{13}+W_{23}\right). \quad (3.2)$$

whith $W_{ij} = \frac{\partial^2 W}{\partial I_i \partial I_j}$, $(i,j = 1,2,3).$

With (3.2), equation (3.1 b) becomes:

$$2\ddot{w}\left[W_{11}+3W_{12}+W_{13}+2W_{22}+W_{33}\right]-\frac{\dot{w}}{R}W_2 = 0. \quad (3.3)$$

Moreover, with the condition $r = R$, (3.1 a) gives $R\dot{w}\left(W_1+W_2\right)=C_0$, where C_0 an integration constant. Thus, considering the condition (2.8 b), the equality (3.1 a) becomes:

$$\dot{w}\left(W_1+W_2\right)=\frac{R_e\sigma_0}{2R}. \quad (3.4)$$

Thus the necessary conditions [8] for an antiplane shear to be possible, are reduced to equalities (3.3) and (3.4). It appears that these conditions strongly relate to the (2.11) energy function W.

4. Case of certain materials

4.1 Diouf-Zidi's model

Consider the energy function [9]:

$$W = a_1\left(I_1-3\right)+a_2\left(I_2-3\right)+a_3\left[\left(I_3^{1/p}-1\right)^p+\left(2-p\right)\left(I_3-1\right)\right]+a_4\frac{2-p}{1+p}\log\left(I_3\right), \quad (4.1)$$

where p is a positive real. (3.1 a)

In order for the shear to take place, the material of type (4.1) must satisfy (3.3) and (3.1 b) (3.4).

Equation (3.4) gives:

$$w(R) = \frac{R_e\sigma_0}{2(a_1+a_2)}\log(R)+w_0,$$

So where the constant w_0 is given by the condition (2.8 a).

By taking (4.2) in (3.3), we obtain a necessary condition translated by the equation:

$$a_3\left[1-2p^2+2(p^2-1)\left[1-\left(\left(\frac{R_e\sigma_0}{2(a_1+a_2)}\right)^{2/p}-1\right)^{p-2}\right]\left(\frac{R_e\sigma_0}{2(a_1+a_2)}\right)^{2/p}\right]\left(\frac{R_e\sigma_0}{2(a_1+a_2)}\right)^{2/p}+$$
$$2a_4p(2-p)-a_2p(1+p)\left(\frac{R_e\sigma_0}{2(a_1+a_2)}\right)^4=0. \quad (4.3)$$

When $p=1$, (respectively $p=2$), (4.1) becomes an Ogden's model (respectively of Hadamard's model), [9] and (4.3) gives a condition on the antiplanar shear for such materials.

4.2 Blatz-Ko 's model

Consider the energy function [10]:

$$W=\frac{\mu f}{2}\left[I_1-1-\frac{1}{\upsilon}+\frac{1-2\upsilon}{\upsilon}I_3^{-\frac{\upsilon}{1-2\upsilon}}\right]+\frac{\mu(1-f)}{2}\left[\frac{I_2}{I_3}-1-\frac{1}{\upsilon}+\frac{1-2\upsilon}{\upsilon}I_3^{\frac{\upsilon}{1-2\upsilon}}\right],$$

the solution of equation (3.4) has the same form as (4.2), it is given by:

$$w(R)=\frac{2R_e\sigma_0}{\mu^2 f(1-f)}\log(R)+w_0$$

It should be noted here that the limiting cases $f=0$ and $f=1$ whose models have been discussed [11] are excluded in this study because of the regularity of the $w(R)$ function.

On the Blatk-Ko's model, the condition (3.3) gives us:

$$2(1-f)(3+\dot{w}^2)+\frac{1-f}{2}+\frac{(1-\upsilon)f}{1-2\upsilon}+\frac{(3\upsilon-1)(1-f)}{1-2\upsilon}=0.$$

Solving this equation we obtain:

$$w(R)=\pm\left\{-3+\frac{1}{2(f-1)}\left[\frac{1-f}{2}+\frac{(1-\upsilon)f}{1-2\upsilon}+\frac{(3\upsilon-1)(1-f)}{1-2\upsilon}\right]\right\}^{1/2}R+w_0.$$

so that the conditions (3.3) and (3.4) are verified for Blatz-Ko'material, it is necessary that the (4.5) and (4.7) are equal. This is only possible when $w(R)$ is a constant, which is $\sigma_0=0,\ f=\dfrac{20\upsilon-11}{18\upsilon-9}$.

4.2 Knowles-Sternberg 's model

As a material model, consider the Knowles-Sternberg's energy density [12]:

$$W=\frac{\mu}{n}\left[1+\frac{b}{n}(I_1-3)\right]^n, \qquad (4.8)$$

and according to the power n, the local equations of motion are of elliptic nature $(n\succ 1/2)$, parabolic $(n=1/2)$ or elliptic-hyperbolic $(n\leq 1/2)$.

The shape of the model (4.8) makes it possible to have a simplified formulation of (3.3) namely: $\ddot{w}W_{11}=0$.

Calculating W_{11}, we note that this is only possible if $n=1$ or $\ddot{w}=0$. However, for the second condition (3.4) necessary for an antiplan movement, we have:

(4.4)

$$\dot{w}=\frac{R_e\sigma_0}{2R}\cdot\frac{n}{\mu b}\left[1+\frac{b}{n}\dot{w}^2\right]^{1-n}. \qquad (4.9)$$

When $n=1$, the variation of $w(R)$ is the same as (4.2) or (4.5). On the other hand, (4.5) because of the variation of $R\in[R_i,R_e]$, the case $\ddot{w}=0$, causes only one possibility for (4.9): $\sigma_0=0$, the same of Blatz-Ko'smodel.

In general, assuming $W(I_1,I_2,I_3)$ linear with respect to I_1 and I_2 [8], we have:

$$W=H_1(I_3)(I_1-3)+H_2(I_3)(I_2-3)+H_3(I_3). \qquad (4.10)$$

(4.6)

For shearing with this material, its energy potential must satisfy conditions (3.3) and (3.4).

(4.7)

Asking $\dfrac{\partial H_i}{\partial R}=\dfrac{\partial H_i}{\partial I_3}\dfrac{\partial I_3}{\partial R}=H_i'\dot{I}_3$,with (3.3) and (3.4) we obtain:

$$2\ddot{w}\left(H_1'+H_3'\right)-\frac{\dot{w}}{R}H_2=0,$$
$$\dot{w}\left(H_1+H_2\right)=\frac{R_e\sigma_0}{2R}. \qquad (4.11)$$

By combining these two equations, we arrive at a single condition for the antiplan transformation to take place:

$$\left(H_1'+H_3''\right)\left(\frac{H_1+H_2+R\dot{I}_3\left(H_1'+H_2'\right)}{H_1+H_2}\right)+\frac{H_2}{2}=0. \qquad (4.12)$$

As in the previous cases, we notice that the deformations translated explicitly by $w(R)$ are logarithmic functions. On the other hand, it should be noted that the condition for a material to undergo an anti-planar shear depends strongly on its potential.

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